Flow of an elastico-viscous fluid between torsionally oscillating disks

By K. R. FRATER
Department of Applied Mathematics, University College of Swansea

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The flow of an incompressible elastico-viscous fluid between two parallel, infinite disks is investigated when one disk is held at rest and the other performs rotary oscillations about their common axis. It is found that the purely periodic primary motion has associated with it a secondary steady velocity distribution, as well as a secondary periodic motion with twice the frequency of the primary. The steady component of the secondary flow is discussed in detail.

1. Introduction

In a recent paper, Rosenblat (1960) examined the flow of a Newtonian fluid between two parallel, infinite disks performing torsional oscillations. Two specific cases were studied: (i) when one disk performs (fairly small) torsional oscillations and the other is at rest, (ii) when both disks oscillate with the same amplitude and frequency and with a phase difference of 180 °. Rosenblat found that solutions satisfying the Navier–Stokes equations and the boundary conditions could be obtained by assuming that certain non-linear terms could be neglected. It was found that the oscillatory rotational motion of the disks induced a radial-axial secondary flow, which had a mean steady component as well as a fluctuating component. Similar effects have more recently been demonstrated theoretically by Bhatnagar & Rajeswari (1962), and Srivastava (1963). Both of these papers were concerned with a special case of the Rivlin–Ericksen 'second-order' fluid.

The purpose of the present paper is to investigate the flow of a fluid with marked transient elasticity of shape. It is to be expected that a fluid with elastic properties will behave somewhat differently from any inelastic viscous fluid when subjected to any kind of oscillatory flow. Also, under the conditions considered by Rosenblat, because of the rotational character of the motion, there is the possibility of a sort of Weissenberg effect taking place. Hence it is of interest to study the effect of torsional oscillations on an elastico-viscous fluid contained between a pair of parallel discs.

The idealized incompressible elastico-viscous fluid considered here has the following equations of state relating the stress tensor S_{ik} and the rate-of-strain tensor $E_{ik} = \frac{1}{2}(U_{k,i} + U_{i,k})$:

$$S_{ik} = P_{ik} - Pg_{ik}, \tag{1}$$

$$P^{ik} + \lambda_1 \frac{\mathfrak{d} P^{ik}}{\mathfrak{d} T} + \mu_0 P^i_j E^{jk} = 2\eta_0 \left(E^{ik} + \lambda_2 \frac{\mathfrak{d} E^{ik}}{\mathfrak{d} T} \right). \tag{2}$$

Here U_i denotes the velocity vector, g_{ik} the metric tensor, P_{ik} the part of the stress tensor related to change of shape of a material element, and P an isotropic pressure; η_0 is a constant having the dimensions of viscosity and λ_1, λ_2, μ_0 are constants having the dimension of time. The derivative $\mathfrak{d}/\mathfrak{d}T$ is the convected time-derivative (Oldroyd 1950) defined thus: if B^{ik} is any contravariant tensor, then we have

$$\frac{\mathfrak{d}B^{ik}}{\mathfrak{d}T} = \frac{\partial B^{ik}}{\partial T} + U^j B^{ik}_{,j} + \Omega^{i}_{.m} B^{mk} + \Omega^{k}_{.m} B^{im} - E^i_m B^{mk} - E^k_m B^{im},$$

where $\Omega_{ik} = \frac{1}{2}(U_{k,i} - U_{i,k})$ is the vorticity tensor.

Oldroyd (1958) has shown that there are idealized fluids of the class defined by equations (1) and (2) which in theory exhibit the following non-Newtonian flow properties that have been observed in polymer solutions and some other elastico-viscous fluids. They have a variable apparent viscosity in simple shearing, decreasing with increasing rate of strain from a limiting value η_0 at low rates to a limiting value η_1 $(= \eta_0\lambda_2/\lambda_1 < \eta_0)$ at high rates; they exhibit the Weissenberg climbing effect, and have a distribution of normal stresses associated with an extra tension along the streamlines in many types of steady simple shearing flow, having an isotropic state of stress in planes normal to the streamlines. For flow at small shear rates, the fluids are characterized by three constants, a coefficient of viscosity η_0, a relaxation time λ_1, and a retardation time λ_2. If the above properties are to be represented by the equations of state over the whole range of rates of shear, Oldroyd found that the constants in equation (2) must satisfy the relations

$$\eta_0 > 0, \quad \lambda_1 > \lambda_2 \geqslant \tfrac{1}{9}\lambda_1 > 0, \quad \mu_0 > 0.$$

Equation (2) is the simplest possible equation of state describing an elastico-viscous fluid with the above properties and has been chosen mainly for mathematical convenience.

In the present paper we shall consider only case (i) of the boundary conditions treated by Rosenblat. Our method of analysis is slightly different from that of Rosenblat, in that we shall expand quantities in powers of Ω, where Ω is the angular amplitude of the motion of the disk.

2. Equations of motion

Consider a mass of elastico-viscous fluid, which is characterized by the equations of state (1), (2), bounded by two parallel disks which are represented by the planes $Z = 0$, $Z = a$ in a cylindrical-polar co-ordinate system (R, θ, Z). The axis $R = 0$, perpendicular to the disks, is taken to be an axis of symmetry for the whole motion. We suppose the physical components of the velocity vector are U, V, W in this system of co-ordinates.

If the disk $Z = 0$ performs oscillations about the axis $R = 0$, with frequency $n/2\pi$ and angular amplitude Ω, while the disk $Z = a$ remains at rest, then the boundary conditions are

$$\left.\begin{aligned} &U = W = 0, \quad V = n\Omega R e^{inT} \quad \text{on} \quad Z = 0, \\ &U = W = V = 0 \quad \text{on} \quad Z = a. \end{aligned}\right\} \tag{3}$$

The convention is adopted that real parts are to be understood whenever complex expressions are quoted for physical quantities.

Equations (2), the usual equations of motion and continuity and the boundary conditions (3), are first reduced to non-dimensional form by the following substitutions:

$$R = ar, \quad Z = az, \quad T = n^{-1}t,$$
$$P_{(ik)} = \eta_0 n p_{(ik)}, \quad P = \rho n^2 a^2 p,$$
$$U = anu, \quad W = anw, \quad V = an\Omega v,$$
$$\lambda_2 = \sigma\lambda_1, \quad \mu_0 = \epsilon\lambda_1,$$

where P_{ik} denotes the physical components of the partial stress tensor and σ, ϵ are clearly two positive dimensionless physical constants of the material. From this point the dimensionless form of the physical components of the partial stress tensor will be denoted by p_{rr}, $p_{\theta z}$, etc. We then obtain the following set of ten equations relating six components of partial stress, three components of velocity and an isotropic pressure:

$$p_{rr} + S\left[\frac{\partial p_{rr}}{\partial t} + u\frac{\partial p_{rr}}{\partial r} + w\frac{\partial p_{rr}}{\partial z} - 2\frac{\partial u}{\partial r}p_{rr} - 2\frac{\partial u}{\partial z}p_{rz}\right] + \epsilon S\frac{\partial u}{\partial r}[p_{rr}+p_{\theta\theta}+p_{zz}]$$
$$= 2\frac{\partial u}{\partial r} + \sigma S\left[2\frac{\partial^2 u}{\partial t\,\partial r} + 2u\frac{\partial^2 u}{\partial r^2} + 2w\frac{\partial^2 u}{\partial z\,\partial r} - 4\left(\frac{\partial u}{\partial r}\right)^2 - 2\frac{\partial u}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)\right], \tag{4}$$

$$p_{\theta\theta} + S\left[\frac{\partial p_{\theta\theta}}{\partial t} + u\frac{\partial p_{\theta\theta}}{\partial r} + w\frac{\partial p_{\theta\theta}}{\partial z} - 2\frac{u}{r}p_{\theta\theta} - 2\Omega r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)p_{r\theta} - 2\Omega\frac{\partial v}{\partial z}p_{\theta z}\right] + \epsilon S\frac{u}{r}[p_{rr}+p_{\theta\theta}+p_{zz}]$$
$$= 2\frac{u}{r} + \sigma S\left[\frac{2}{r}\frac{\partial u}{\partial t} + 2u\left(\frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) + 2\frac{w}{r}\frac{\partial u}{\partial z} - 4\left(\frac{u}{r}\right)^2 - 2\Omega^2 r^2\left\{\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right\}^2 - 2\Omega^2\left(\frac{\partial v}{\partial z}\right)^2\right], \tag{5}$$

$$p_{zz} + S\left[\frac{\partial p_{zz}}{\partial t} + u\frac{\partial p_{zz}}{\partial r} + w\frac{\partial p_{zz}}{\partial z} - 2\frac{\partial w}{\partial z}p_{zz} - 2\frac{\partial w}{\partial r}p_{rz}\right] + \epsilon S\frac{\partial w}{\partial z}[p_{rr}+p_{\theta\theta}+p_{zz}]$$
$$= 2\frac{\partial w}{\partial z} + \sigma S\left[2\frac{\partial^2 w}{\partial t\,\partial z} + 2u\frac{\partial^2 w}{\partial r\,\partial z} + 2w\frac{\partial^2 w}{\partial z^2} - 4\left(\frac{\partial w}{\partial z}\right)^2 - 2\frac{\partial w}{\partial r}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)\right], \tag{6}$$

$$p_{\theta z} + S\left[\frac{\partial p_{\theta z}}{\partial t} + u\frac{\partial p_{\theta z}}{\partial r} + w\frac{\partial p_{\theta z}}{\partial z} - \left(\frac{\partial w}{\partial r}+\frac{u}{r}\right)p_{\theta z} - \frac{\partial w}{\partial r}p_{r\theta} - \Omega r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)p_{rz} - \Omega\frac{\partial v}{\partial z}p_{zz}\right] + \tfrac{1}{2}\epsilon\Omega S\frac{\partial v}{\partial z}[p_{rr}+p_{\theta\theta}+p_{zz}]$$
$$= \Omega\left\{\frac{\partial v}{\partial z} + \sigma S\left[\frac{\partial^2 v}{\partial t\,\partial z} + u\frac{\partial^2 v}{\partial r\,\partial z} + w\frac{\partial^2 v}{\partial z^2} - \left(\frac{\partial w}{\partial r}+\frac{u}{r}\right)\frac{\partial v}{\partial z} - \left(2\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right)\frac{r\partial}{\partial r}\left(\frac{v}{r}\right) - 2\frac{\partial v}{\partial z}\frac{\partial w}{\partial z}\right]\right\}, \tag{7}$$

$$p_{rz}+S\left[\frac{\partial p_{rz}}{\partial t}+u\frac{\partial p_{rz}}{\partial r}+w\frac{\partial p_{rz}}{\partial z}-\left(\frac{\partial u}{\partial r}+\frac{\partial w}{\partial z}\right)p_{rz}-\frac{\partial w}{\partial r}p_{rr}-\frac{\partial u}{\partial z}p_{zz}\right]$$
$$+\tfrac{1}{2}\epsilon S\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)[p_{rr}+p_{\theta\theta}+p_{zz}]$$
$$=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}+\sigma S\left[\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)+u\left(\frac{\partial^2 u}{\partial r\partial z}+\frac{\partial^2 w}{\partial r^2}\right)+w\left(\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial z\partial r}\right)\right.$$
$$\left.-2\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}-\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)\left(\frac{\partial u}{\partial r}+\frac{\partial w}{\partial z}\right)-2\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\right], \tag{8}$$

$$p_{r\theta}+S\left[\frac{\partial p_{r\theta}}{\partial t}+u\frac{\partial p_{r\theta}}{\partial r}+w\frac{\partial p_{r\theta}}{\partial z}-\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)p_{r\theta}-\frac{\partial u}{\partial z}p_{\theta z}-\Omega\frac{r\partial}{\partial r}\left(\frac{v}{r}\right)p_{rr}-\Omega\frac{\partial v}{\partial z}p_{rz}\right]$$
$$+\tfrac{1}{2}\epsilon\Omega Sr\frac{\partial}{\partial r}\left(\frac{v}{r}\right)[p_{rr}+p_{\theta\theta}+p_{zz}]$$
$$\Omega=\left\{r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)+\sigma S\left[r\frac{\partial^2}{\partial t\partial r}\left(\frac{v}{r}\right)+u\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right\}+wr\frac{\partial^2}{\partial z\partial r}\left(\frac{v}{r}\right)-\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right.\right.$$
$$\left.\left.-r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)-2r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\frac{\partial u}{\partial r}-\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)\frac{\partial v}{\partial z}\right]\right\}, \tag{9}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\Omega^2\frac{v^2}{r}=-\frac{\partial p}{\partial r}+\frac{1}{R_0}\left\{\frac{1}{r}\frac{\partial}{\partial r}[rp_{rr}]+\frac{\partial}{\partial z}p_{rz}-\frac{p_{\theta\theta}}{r}\right\}, \tag{10}$$

$$\Omega\left[\frac{\partial v}{\partial t}+\frac{u}{r}\frac{\partial}{\partial r}(rv)+w\frac{\partial v}{\partial z}\right]=\frac{1}{R_0}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}[r^2p_{r\theta}]+\frac{\partial}{\partial z}p_{\theta z}\right\}, \tag{11}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\frac{1}{R_0}\left\{\frac{1}{r}\frac{\partial}{\partial r}[rp_{rz}]+\frac{\partial p_{zz}}{\partial z}\right\}, \tag{12}$$

$$\frac{\partial}{\partial r}(ru)+\frac{\partial}{\partial z}(rw)=0, \tag{13}$$

where $R_0=\rho na^2/\eta_0$ is a Reynolds number for the flow, and $S=\lambda_1 n$ can be thought of as a dimensionless measure of the 'memory' of the elastico-viscous fluid, based on the use of the period of oscillation as the natural unit of time. The associated boundary conditions are

$$\left.\begin{aligned}&u=w=0,\quad v=r\,e^{it}\quad\text{on}\quad z=0,\\&u=v=w\quad\text{on}\quad z=1.\end{aligned}\right\} \tag{14}$$

3. Solution of the equations

In order to obtain an approximate solution of the above equations, it is now assumed that Ω is sufficiently small so that we may expand quantities in the form of power series in Ω. Then, from the form of the equations (4)–(13) and the boundary conditions (14), we shall expect u, v, w and the p_{ik} to be given by

$$v=rf(z)\,e^{it}+\Omega^2f_1(r,z,t)+\ldots,\qquad u=\Omega^2g(r,z,t)+\ldots,\qquad w=\Omega^2h(r,z,t)+\ldots,$$
$$p_{rr}=\Omega^2G(r,z,t)+\ldots,\qquad p_{zz}=\Omega^2H(r,z,t)+\ldots,\quad p_{\theta\theta}=\Omega^2K(r,z,t)+\ldots,$$
$$p_{\theta z}=\Omega rF(z)\,e^{it}+\Omega^3F_1(r,z,t)+\ldots,\quad p_{r\theta}=\Omega^3M(r,z,t)+\ldots,\quad p_{rz}=\Omega^2L(r,z,t)+\ldots,$$
$$p=\Omega^2N(r,z,t)+\ldots.$$

If these expressions are substituted in equations (4)–(13) and the boundary conditions (14), and coefficients of Ω, Ω^2, etc., are equated, the following system of linear partial differential equations is obtained:

$$G+S\frac{\partial G}{\partial t} = 2\frac{\partial g}{\partial r}+2\sigma S\frac{\partial^2 g}{\partial t\partial r}, \tag{15}$$

$$K+S\frac{\partial K}{\partial t}-2Sr^2(f'e^{it})(Fe^{it}) = \frac{2g}{r}+2\sigma S\frac{\partial}{\partial t}\left(\frac{g}{r}\right)-2\sigma Sr^2(f'e^{it})^2, \tag{16}$$

$$H+S\frac{\partial H}{\partial t} = 2\frac{\partial h}{\partial z}+2\sigma S\frac{\partial^2 h}{\partial t\partial z}, \tag{17}$$

$$L+S\frac{\partial L}{\partial t} = \frac{\partial g}{\partial z}+\frac{\partial h}{\partial r}+\sigma S\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial z}+\frac{\partial h}{\partial r}\right), \tag{18}$$

$$(1+iS)F = (1+i\sigma S)f', \tag{19}$$

$$\frac{\partial g}{\partial t}-r(fe^{it})^2 = -\frac{\partial N}{\partial r}+\frac{1}{R_0}\left[\frac{1}{r}\frac{\partial}{\partial r}(rG)+\frac{\partial L}{\partial z}-\frac{K}{r}\right], \tag{20}$$

$$\frac{\partial h}{\partial t} = -\frac{\partial N}{\partial z}+\frac{1}{R_0}\left[\frac{1}{r}\frac{\partial}{\partial r}(rL)+\frac{\partial H}{\partial z}\right], \tag{21}$$

$$iR_0 f = F', \tag{22}$$

$$\frac{\partial}{\partial r}(rg)+\frac{\partial}{\partial z}(rh) = 0, \tag{23}$$

where a prime denotes differentation with respect to z. The boundary conditions become

$$\left.\begin{aligned} f=0,\quad g=0,\quad h=0 \quad \text{on} \quad z=1,\\ f=1,\quad g=0,\quad h=0 \quad \text{on} \quad z=0.\end{aligned}\right\} \tag{24}$$

The primary flow

Eliminating F from equations (19) and (22) gives the following ordinary differential equation for f:

$$f'' = -\alpha^2 R_0 f, \tag{25}$$

where

$$\alpha^2 = -i(1+iS)/(1+i\sigma S). \tag{26}$$

The solution of equation (25) that satisfies the boundary conditions (24) is

$$f = \sin[\alpha R_0^{\frac{1}{2}}(1-z)]/\sin\alpha R_0^{\frac{1}{2}}. \tag{27}$$

Also, from equation (19), we obtain

$$F = iR_0^{\frac{1}{2}}\cos[\alpha R_0^{\frac{1}{2}}(1-z)]/\alpha\sin\alpha R_0^{\frac{1}{2}}. \tag{28}$$

Hence, with an error of the order of Ω^2, we have in real terms

$$\frac{v}{r} = \frac{\begin{aligned}&[\{\cos(|\alpha|R_0^{\frac{1}{2}}z\cos\chi)\cosh[|\alpha|R_0^{\frac{1}{2}}(2-z)\sin\chi]\\&\quad-\cos[|\alpha|R_0^{\frac{1}{2}}(2-z)\cos\chi]\cosh(|\alpha|R_0^{\frac{1}{2}}z\sin\chi)\}\cos t\\&+\{\sin[|\alpha|R_0^{\frac{1}{2}}(2-z)\cos\chi]\sinh(|\alpha|R_0^{\frac{1}{2}}z\sin\chi)\\&\quad-\sin(|\alpha|R_0^{\frac{1}{2}}z\cos\chi)\sinh[|\alpha|R_0^{\frac{1}{2}}(2-z)\sin\chi]\}\sin t]\end{aligned}}{\cosh(2|\alpha|R_0^{\frac{1}{2}}\sin\chi)-\cos(2|\alpha|R_0^{\frac{1}{2}}\cos\chi)}, \tag{29}$$

where, from equation (26),

$$|\alpha|^2 = \{(1+S^2)/(1+\sigma^2 S^2)\}^{\frac{1}{2}}, \quad 2\chi = -\tfrac{1}{2}\pi + \tan^{-1} S - \tan^{-1} \sigma S.$$

For large R_0, (29) becomes

$$v/r \sim \exp(-|\alpha| R_0^{\frac{1}{2}} z \sin\phi) \cos(t - |\alpha| R_0^{\frac{1}{2}} z \cos\phi), \tag{30}$$

where $\phi = -\chi$, $0 \leqslant \phi \leqslant \frac{1}{4}\pi$. In the case of a viscous fluid, Rosenblat has shown that (30) is a valid approximation to (29) for $R_0 > 20$. Since we are more interested in high-frequency oscillations, i.e. large R_0 for given a, η_0 and ρ, the asymptotic form of (29) would be quite suitable for the purpose of computation.

The secondary flow

Using the fact that, if z_1, z_2 are any two complex numbers,

$$(\operatorname{Re} z_1)(\operatorname{Re} z_2) = \tfrac{1}{2}\operatorname{Re}(z_1 \bar{z}_2) + \tfrac{1}{2}\operatorname{Re}(z_1 z_2),$$

we have

$$(fe^{it})^2 = \tfrac{1}{2} f\bar{f} + \tfrac{1}{2} f^2 e^{2it}, \quad (f'e^{it})^2 = \tfrac{1}{2} f'\bar{f}' + \tfrac{1}{2} f'^2 e^{2it}, \quad (f'e^{it})(Fe^{it}) = \tfrac{1}{2} f'\bar{F} + \tfrac{1}{2} f'Fe^{2it}, \tag{31}$$

where $\bar{f}$ denotes the complex conjugate of f, etc. Hence from equations (15)–(23) we expect G, K, H, g and h to be of the form

$$\left.\begin{aligned} G &= G_1(r,z) + G_2(r,z)\, e^{2it}, & K &= K_1(r,z) + K_2(r,z)\, e^{2it}, \\ H &= H_1(r,z) + H_2(r,z)\, e^{2it}, & L &= L_1(r,z) + L_2(r,z)\, e^{2it}, \\ g &= g_1(r,z) + g_2(r,z)\, e^{2it}, & h &= h_1(r,z) + h_2(r,z)\, e^{2it}, \end{aligned}\right\} \tag{32}$$

and, following Rosenblat, we shall assume N to be of the form

$$N = N_1(z,t) + \tfrac{1}{2} r^2 (N_2 + N_3 e^{2it}), \tag{33}$$

where N_2, N_3 are constants. It follows that the purely periodic primary motion has associated with it an additional steady velocity distribution, as well as a periodic motion with twice the frequency of the primary. We shall consider, in the present paper, only the steady component of this secondary flow.

Substituting the expressions (32) and (33) into equations (15)–(23) and equating time-independent parts, we obtain the following equations

$$G_1 = 2\,\partial g_1/\partial r, \tag{34}$$

$$K_1 - r^2 S(f'\bar{F}) = 2g_1/r - \sigma r^2 S(f'\bar{f}'), \tag{35}$$

$$H_1 = 2\,\partial h_1/\partial z, \tag{36}$$

$$L_1 = \partial g_1/\partial z + \partial h_1/\partial r, \tag{37}$$

$$-\tfrac{1}{2} r(f\bar{f}) = -rN_2 + \frac{1}{R_0}\left[\frac{1}{r}\frac{\partial}{\partial r}(rG_1) + \frac{\partial L_1}{\partial z} - \frac{K_1}{r}\right], \tag{38}$$

$$\partial(rg_1)/\partial r + \partial(rh_1)/\partial z = 0. \tag{39}$$

Equations (34)–(39) can clearly be satisfied by writing

$$g_1 = r\,d\psi/dz, \quad h_1 = -2\psi, \tag{40}$$

where $\psi = \psi(z)$. Eliminating G_1, L_1 and K_1 from equations (34)–(39) and using (40), we obtain the following ordinary differential equation for ψ

$$d^3\psi/dz^3 = R_0 N_2 - \tfrac{1}{2}R_0 f\bar{f} + S(f'\bar{F} - \sigma f'\bar{f}'). \tag{41}$$

This equation must be solved subject to the boundary conditions

$$\psi = 0, \quad \partial\psi/dz = 0 \quad \text{on} \quad z = 0, \quad z = 1. \tag{42}$$

Now, from (27) and (28), we have

$$f\bar{f} = \frac{\cos[2|\alpha|\, R_0^{\frac{1}{2}}(1-z)\cos\chi] - \cosh[2|\alpha|R_0^{\frac{1}{2}}(1-z)\sin\chi]}{\cos(2|\alpha|R_0^{\frac{1}{2}}\cos\chi) - \cosh(2|\alpha|\, R_0^{\frac{1}{2}}\sin\chi)},$$

$$f'\bar{f}' = -|\alpha|^2 R_0 \left[\frac{\cos[2|\alpha|\, R_0^{\frac{1}{2}}(1-z)\cos\chi] + \cosh[2|\alpha|\, R_0^{\frac{1}{2}}(1-z)\sin\chi]}{\cos(2|\alpha|R_0^{\frac{1}{2}}\cos\chi) - \cosh(2|\alpha|R_0^{\frac{1}{2}}\sin\chi)}\right],$$

$$f'\bar{F} = R_0 \sin 2\chi \left[\frac{\cos[2|\alpha|R_0^{\frac{1}{2}}(1-z)\cos\chi] + \cosh[2|\alpha|R_0^{\frac{1}{2}}(1-z)\sin\chi]}{\cos(2|\alpha|\, R_0^{\frac{1}{2}}\cos\chi) - \cosh(2|\alpha|R_0^{\frac{1}{2}}\sin\chi)}\right].$$

Substituting the above expressions into equation (41), we obtain

$$2\lambda^{-2} d^3\psi/dz^3 = N_2 - \tfrac{1}{2}[\cos q\lambda - \cosh b\lambda]^{-1}[\mu_1 \cos q\lambda(1-z) - \mu_2 \cosh b\lambda(1-z)], \tag{43}$$

where $\lambda^2 = 2R_0, \quad q = \sqrt{2}|\alpha|\cos\chi, \quad b = \sqrt{2}|\alpha|\sin\chi,$

$$\mu_1 = 1 - 2S(\sin 2\chi + \sigma|\alpha|^2), \quad \mu_2 = 1 + 2S(\sin 2\chi + \sigma|\alpha|^2).$$

The solution of (43) that satisfies the boundary conditions (40) is given by

$$\begin{aligned}\psi = (\cosh b\lambda - \cos q\lambda)^{-1}\bigg\{&\frac{1}{4\lambda}\left(\frac{\mu_2}{b^3}\sinh b\lambda(1-z) + \frac{\mu_1}{q^3}\sin q\lambda(1-z)\right)\\ &-\frac{1}{4\lambda}\left(\frac{\mu_2}{b^3}\sinh b\lambda + \frac{\mu_1}{q^3}\sin q\lambda\right)(1-3z^2+2z^3) - \frac{1}{4}\left(\frac{\mu_2}{b^2}+\frac{\mu_1}{q^2}\right)z^2(1-z)\\ &+\frac{1}{4}\left(\frac{\mu_2}{b^2}\cosh b\lambda + \frac{\mu_1}{q^2}\cos q\lambda\right)z(1-2z+z^2)\bigg\};\end{aligned} \tag{44}$$

also N_2 is given by

$$\begin{aligned}N_2 = \frac{3}{\lambda^3}(\cosh b\lambda - \cos q\lambda)^{-1}\bigg\{&\left(\frac{\mu^2}{b^2}+\frac{\mu_1}{q^2}\right)\lambda + \left(\frac{\mu_2}{b^2}\cosh b\lambda + \frac{\mu_1}{q^2}\cos q\lambda\right)\\ &-2\left(\frac{\mu_2}{b^3}\sinh b\lambda + \frac{\mu_1}{q^3}\sin q\lambda\right)\bigg\}.\end{aligned} \tag{45}$$

Differentiating (44) with respect to z, we find

$$\begin{aligned}\psi' = (\cosh b\lambda - \cos q\lambda)^{-1}\bigg\{&-\frac{1}{4}\left(\frac{\mu_2}{b^2}\cosh b\lambda(1-z) + \frac{\mu_1}{q^2}\cosh q\lambda(1-z)\right)\\ &+\frac{3}{2\lambda}\left(\frac{\mu_2}{b^3}\sinh b\lambda + \frac{\mu_1}{q^3}\sin q\lambda\right)z(1-z) - \frac{1}{4}\left(\frac{\mu_2}{b^2}+\frac{\mu_1}{q^2}\right)(2z-3z^2)\\ &+\frac{1}{4}\left(\frac{\mu_2}{b^2}\cosh b\lambda + \frac{\mu_1}{q^2}\cos q\lambda\right)(1-4z+3z^2)\bigg\}.\end{aligned} \tag{46}$$

From the equation of continuity and (40) it follows that the streamlines of the steady radial-axial secondary flow are given by

$$r^2\psi = \text{const.} \tag{47}$$

4. Discussion

For computational purposes, we may use the asymptotic expansions of (44), (45) and (46) for $R_0 > 20$, and these give the approximations

$$\psi \sim \tfrac{1}{4}\Phi_1\{z(1-z)^2-[(1+2z)(1-z)^2-\exp(-2\sqrt{R_0}|\alpha|z\sin\phi)]/(2\sqrt{R_0}|\alpha|\sin\phi)\}, \tag{48}$$

$$\psi' \sim \tfrac{1}{4}\Phi_1\{(1-z)(1-3z)+3z(1-z)/(\sqrt{R_0}|\alpha|\sin\phi)-\exp(-2\sqrt{R_0}|\alpha|z\sin\phi)\}, \tag{49}$$

$$N_2 \sim (3\Phi_1/2R_0)(1-1/\sqrt{R_0}|\alpha|\sin\phi), \tag{50}$$

where

$$\Phi_1 = [1-2S(\sin 2\phi-\sigma|\alpha|^2)]/(2|\alpha|^2\sin^2\phi). \tag{51}$$

Now, from equation (26) we obtain

$$2\phi = \tfrac{1}{2}\pi-\tan^{-1}S+\tan^{-1}\sigma S,$$

and it follows that

$$\cot 2\phi = \frac{(1-\sigma)S}{1+\sigma S^2}, \quad \sin 2\phi = \frac{1+\sigma S^2}{(1+S^2)^{\frac{1}{2}}(1+\sigma^2S^2)^{\frac{1}{2}}};$$

hence $\quad 1-2S(\sin 2\phi-\sigma|\alpha|^2) = 1-2S(1-\sigma)/[(1+S^2)^{\frac{1}{2}}(1+\sigma^2S^2)^{\frac{1}{2}}].$

From (51) we find that Φ_1 will vanish when

$$\sigma^2S^4-(3-8\sigma+3\sigma^2)S^2+1 = 0, \tag{52}$$

and for this equation for S to have real roots, we must have

$$(3-8\sigma+3\sigma^2)^2 \geqslant 4\sigma^2;$$

since σ is a positive number less than unity,

$$\sigma \leqslant \tfrac{1}{3}. \tag{53}$$

Hence we find that, if $\sigma \leqslant \frac{1}{3}$, there exist certain real values of S that will make Φ_1 vanish; and when Φ_1 does vanish, equations (48), (49) and (50) give an inadequate approximation and must be replaced by

$$\psi = \Phi_2\exp(-2\sqrt{R_0}|\alpha|\sin\phi)\left\{z(1-z)\left(\frac{\cos q\lambda}{1+\cos q\lambda}-z\right)(1+\cos q\lambda)\right.$$
$$\left.+\frac{\sin q\lambda(1-z)-(1-z)^2(1+2z)\sin q\lambda}{q\lambda}\right\}, \tag{54}$$

$$\psi' = \tfrac{1}{4}\Phi_2\exp(-2\sqrt{R_0}|\alpha|\sin\phi)[(1-3z)(1-z)\cos q\lambda-z(2-3z)$$
$$-\cos q\lambda(1-z)+(6/q\lambda)z(1-z)\sin q\lambda], \tag{55}$$

$$N_2 = 3(\Phi_2/\lambda^3)\exp(-2\sqrt{R_0}\sin\phi)[\lambda+\lambda\cos q\lambda-2\sin q\lambda/q], \tag{56}$$

where

$$\Phi_2 = \{1+2S(\sin 2\phi-\sigma|\alpha|^2)\}/2|\alpha|^2\cos^2\phi. \tag{57}$$

When $\sigma < \frac{1}{3}$, Φ_1 vanishes for two distinct values of S; solving (52), we find the critical values of S are

$$S_1, S_2 = \left[\frac{(3-8\sigma+3\sigma^2)\mp\sqrt{[(3-8\sigma+3\sigma^2)^2-4\sigma^2]}}{2\sigma^2}\right]^{\frac{1}{2}}.$$

The way in which Φ_1 varies with S in the elastico-viscous cases $\sigma = \frac{1}{9}, \frac{1}{4}$ and the Newtonian case ($\sigma = 1$), is illustrated in figure 1.

We now see that, if $\sigma < \frac{1}{3}$, the direction of the steady component of the secondary flow behaves as follows. When $S < S_1$, the flow is in a positive sense (if we agree to describe the direction of flow in the case of a Newtonian fluid as in a positive sense). When $S_1 < S < S_2$, the flow is in a negative sense, i.e. the flow is reversed in direction. When $S_2 < S$, the flow is in a positive sense again. As S tends to infinity, the flow becomes identical with that of a Newtonian fluid with a coefficient of viscosity $\lambda_2\eta_0/\lambda_1$. If $\sigma > \frac{1}{3}$, the direction of flow does not change over the whole range of values of S.

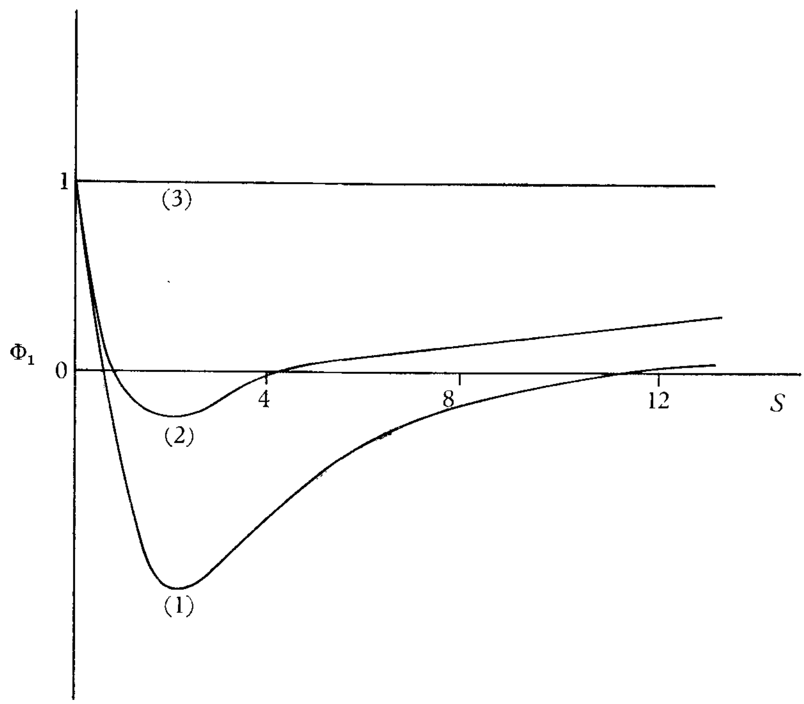

FIGURE 1. The variation of Φ_1 with S, showing critical values of S at which the steady secondary flow changes direction. (1) Elastico-viscous fluid with $\sigma = \frac{1}{9}$. (2) Elastico-viscous fluid with $\sigma = \frac{1}{4}$. (3) Newtonian fluid ($\sigma = 1$).

The solutions (48) and (49) are given in graphical form. Figure 2 shows the dimensionless radial velocity ψ' of the steady secondary flow in the elastico-viscous cases $\sigma = \frac{1}{9}, \frac{1}{4}$ and the Newtonian case ($\sigma = 1$), under each of the conditions (*a*) $R_0 = 25$, $S = 0{\cdot}5$, (*b*) $R_0 = 100$, $S = 2$, (*c*) $R_0 = 500$, $S = 10$. Comparing these particular elastico-viscous fluids with the Newtonian fluid, we observe that, as predicted above, if $S < S_1$ or $S > S_2$, only the magnitude of ψ' is changed as σ decreases; but if $S_1 < S < S_2$, the direction of ψ' is also changed.

Figure 3 depicts, schematically, typical streamlines ($r^2\psi = 0{\cdot}02$) in each of the elastico-viscous cases $\sigma = \frac{1}{9}, \frac{1}{4}$ and the Newtonian case ($\sigma = 1$) under the conditions $R_0 = 500$, $S = 10$. In the Newtonian case and the elastico-viscous case $\sigma = \frac{1}{4}$, fluid is expelled near the disc $z = 0$ and drawn in near the disk $z = 1$, but in the elastico-viscous case $\sigma = \frac{1}{9}$, the reverse effect takes place. The general shape of the streamlines is only slightly affected by the elastic properties of the fluid, even though the direction of flow is strongly dependent on the magnitudes of σ and S.

We conclude that for certain values of the elastic constant σ, namely $\sigma < \frac{1}{3}$, there is a critical range of values of S in which the direction of the steady secondary flow is reversed compared with that in a Newtonian fluid. The predicted reversal phenomenon is a sort of Weissenberg effect and it clearly arises because

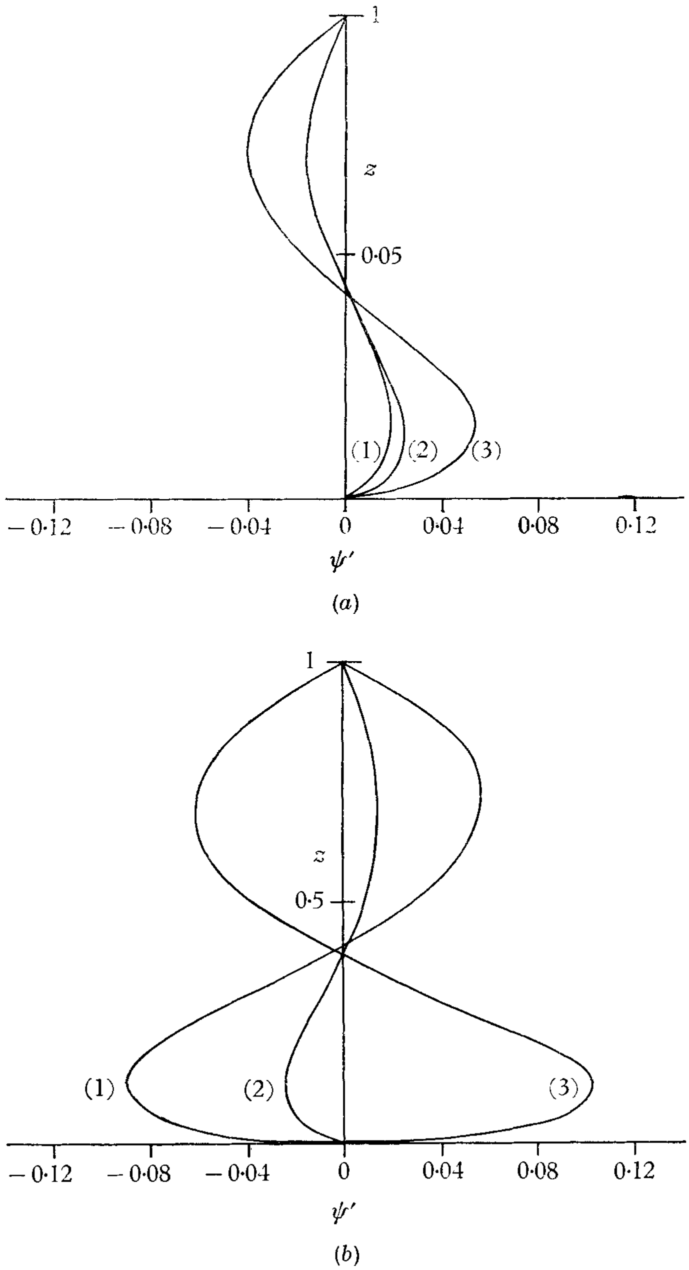

FIGURE 2. For legend see p. 185.

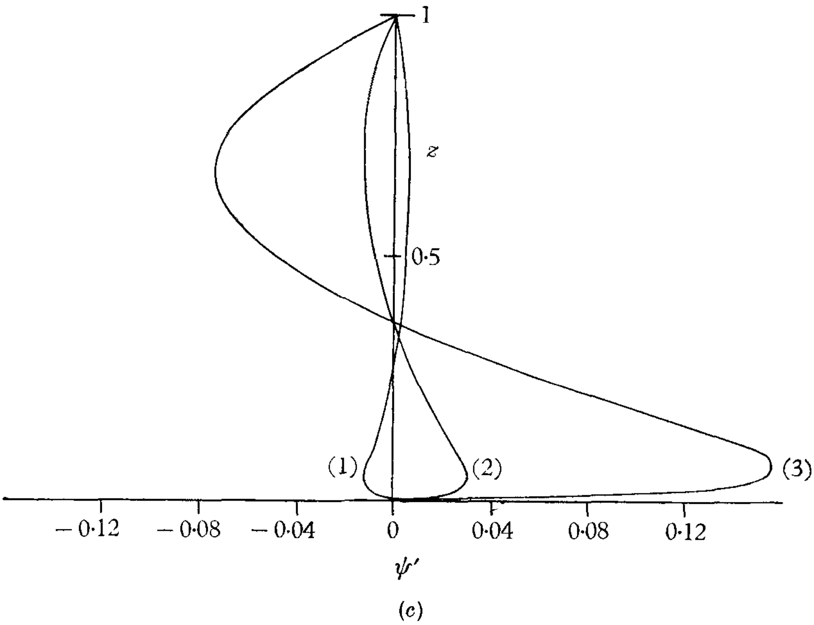

FIGURE 2. The variation of the dimensionless radial velocity with the dimensionless distance across the gap in the steady secondary flow when (*a*) $R_0 = 25$, $S = 0{\cdot}5$; (*b*) $R_0 = 100$, $S = 2$; (*c*) $R_0 = 500$, $S = 10$. (1) Elastico-viscous fluid with $\sigma = \frac{1}{9}$. (2) Elastico-viscous fluid with $\sigma = \frac{1}{4}$. (3) Newtonian fluid ($\sigma = 1$).

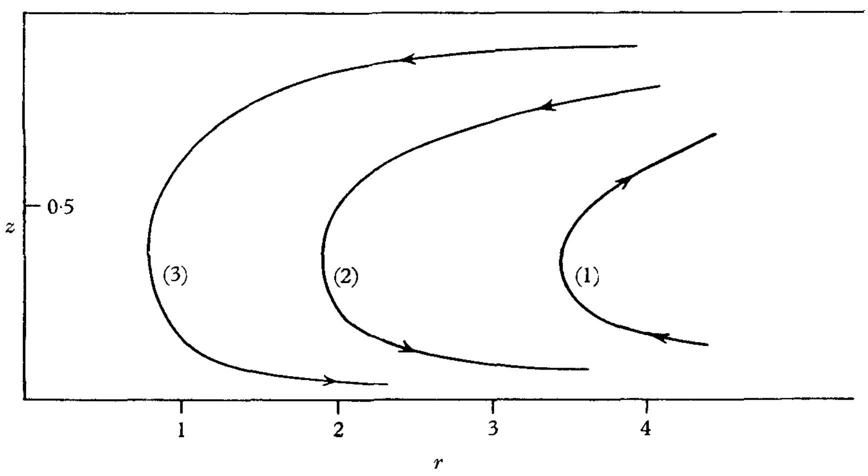

FIGURE 3. Typical streamlines ($r^2\psi = 0{\cdot}02$) of the steady secondary flow when $R_0 = 500$, $S = 10$. (1) Elastico-viscous fluid with $\sigma = \frac{1}{9}$. (2) Elastico-viscous fluid with $\sigma = \frac{1}{4}$. (3) Newtonian fluid ($\sigma = 1$).

of the rotational nature of the motion of the disks. The extra tension along the streamlines induced by shearing remains non-negative throughout each period of oscillation and has the effect of squeezing the fluid in certain regions towards the axis of symmetry. In favourable circumstances this effect overwhelms the expected 'centrifugal force' effects.

Bhatnagar & Rajeswari (1962) found that a similar reversal of the direction of the steady secondary flow is a characteristic feature of the Rivlin–Ericksen fluid. It appears that the main difference between the results of the present paper and those of Bhatnagar & Rajeswari is that these authors showed that it is always possible to find a value of the Reynolds number, above which their type of flow is reversed in direction. It must be remembered that the fluids considered by Bhatnagar & Rajeswari show retarded response to applied stress, but do not show relaxation of stress at constant strain; also the normal stress differences do not correspond to a simple tension along the streamlines. It is not, therefore, surprising that flow of the elastico-viscous fluids considered here should show some distinctive features.

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